Interlacing and non-orthogonality of spectral polynomials for the Lamé operator

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Abstract

Polynomial solutions to the Heine-Stieltjes equation, *the Stieltjes polynomials*, and the associated *Van Vleck polynomials* have been studied since the 1830's in various contexts including the solution of the Laplace equation on an ellipsoid. Recently there has been renewed interest in the distribution of the zeros of Van Vleck polynomials as the degree of the corresponding Stieltjes polynomials increases. In this paper we show that the zeros of Van Vleck polynomials corresponding to Stieltjes polynomials of successive degrees interlace. We also show that the spectral polynomials formed from the Van Vleck zeros are not orthogonal with respect to any measure. This furnishes a counterexample, coming from a second order differential equation, to the well known theorem that the zeros of orthogonal polynomials interlace.

1 Introduction

Let $\alpha_1, \dots, \alpha_n$ be any n distinct complex numbers, and let $\rho_1, \dots, \rho_n$ be positive numbers. The *generalized Lamé equation* is the second order ODE given by

$$\prod_{j=1}^{n}(z-\alpha_j)\,\phi''(z) + 2\sum_{j=1}^{n}\rho_j\prod_{i\neq j}(z-\alpha_i)\,\phi'(z) = V(z)\,\phi(z). \tag{1}$$

According to a result of Heine [?], there exist at most $\sigma(n,k) = \frac{(n+k-2)!}{(n-2)!\,k!}$ polynomials V of degree $n-2$ for which (1) has a polynomial solution ϕ of degree k. These polynomial solutions are often called *Stieltjes polynomials*, and the corresponding polynomials V are known as *Van Vleck polynomials*.

The equation (1) was studied by Lamé in the 1830's in the special case $n = 3$, $\rho_i = 1/2$, $\alpha_1+\alpha_2+\alpha_3 = 0$ in connection with the separation of variables in the Laplace equation using elliptical coordinates [?, Ch. 23]. The equation has since found other applications in studies as diverse as electrostatics and the quantum asymmetric top.

For $\alpha_1, ..., \alpha_n$ real, Stieltjes [**?**] showed that the location of the zeros of the Stieltjes polynomials are completely characterized by their distribution in the subintervals $(\alpha_1, \alpha_2), \ldots, (\alpha_{n-1}, \alpha_n)$. Similar results for the zeros of the Van Vleck polynomials were also obtained by Shah [**?**]. Much is known about the properties of Van Vleck polynomials for a fixed degree of the corresponding Stieltjes polynomial (see, e.g. [**?**] for recent results), but there are few results relating the Van Vleck zeros that correspond to Stieltjes polynomials of different degrees. Recently there has been interest in the distribution of the zeros of Van Vleck polynomials as the degree of the corresponding Stieltjes polynomials tends toward infinity [**?**].

In this paper we consider the case of three α_i's on the real line and first degree Van Vleck polynomials. In this case, let $\alpha_1 < \alpha_2 < \alpha_3$ and $\rho_1, \rho_2, \rho_3 > 0$ be real numbers and define

$$A(x) = \prod_{j=1}^{3} (x - \alpha_j), \quad B(x) = \sum_{j=1}^{3} 2\rho_j \prod_{i \neq j} (x - \alpha_i).$$

Then the Lamé equation is

$$A(x)\phi'' + B(x)\phi' = \mu(x - \nu)\phi. \tag{2}$$

By the Van Vleck zeros of order k we mean the set of all ν's such that (2) has a polynomial solution of degree k. In this case where the α_i's are real, Heine's result is exact, and Van Vleck showed that the $k+1$ Van Vleck zeros of order k are distinct and lie in the interval (α_1, α_3) [**?**].

This paper has two main results. In §2 we show that the Van Vleck zeros of successive orders interlace. That is, if the Van Vleck zeros of order k are written in increasing order as $\nu_1^{(k)} < \nu_2^{(k)} < \cdots < \nu_{k+1}^{(k)}$, then

$$\alpha_1 < \nu_1^{(k+1)} < \nu_1^{(k)} < \nu_2^{(k+1)} < \nu_2^{(k)} < \cdots < \nu_{k+1}^{(k)} < \nu_{k+2}^{(k+1)} < \alpha_3. \tag{3}$$

The proof of this result will be carried out in two steps. First we will show that the Van Vleck zeros of order k and $k+1$ are distinct. Then we will show that the interlacing property (3) holds for a special set of α_i, ρ_i. Since the Van Vleck zeros are continuous functions of these parameters, the interlacing property must hold in general.

Given the interlacing property and the well-known properties of orthogonal polynomials, it is natural to ask whether the polynomials formed from the Van Vleck zeros,

$$\prod_{i=1}^{k+1} \left(x - \nu_i^{(k)}\right),$$

are orthogonal with respect to some measure. These polynomials are often referred to as *spectral polynomials*, as the Van Vleck zeros are often interpreted as an energy. Their study goes back as far as Hermite, and recent results are found in [**?**] and the references therein. In [**?**, **?**] the spectral polynomials are studied in conjunction with the quantum Euler top. The authors of [?], in particular, relate the coefficients of the spectral polynomials to Bernoulli polynomials. In §3 we show that the spectral polynomials are not orthogonal with respect to any measure. We conclude with a few remarks and a conjecture.

2 Van Vleck zeros of successive orders interlace

Our first step is to prove a lemma showing that the Van Vleck zeros of order k and $k+1$ are distinct. We follow a Sturm comparison type argument to argue that if there were a Van Vleck zero of order k and $k+1$ in common, the Stieltjes polynomials corresponding to this common zero would have interlacing zeros, leading to a contradiction. Before we proceed to the lemma we collect some facts about Stieltjes polynomials that will be used throughout.

The first is a result due to Stieltjes [?] (see also [?]), that the zeros of every polynomial solution $S(z)$ of (1) lie in the smallest convex polygon containing $\alpha_1, \ldots, \alpha_n$. This follows from the fact that if $z_1, \ldots, z_l$ are the zeros of a polynomial S satisfying (1), and z_r is not an α_i, then

$$\sum_{j \neq r} \frac{1}{z_r - z_j} + \sum_{j} \frac{\rho_j}{z_r - \alpha_j} = 0.$$

Appealing to the Gauss-Lucas Theorem, z_r must lie in the smallest convex polygon containing $z_1, \ldots, z_{r-1}, z_{r+1}, \ldots, z_l, \alpha_1, \ldots, \alpha_n$. Since this is true for each r, the result follows. Additionally, every zero z_i of S is simple unless z_i corresponds to an α_i, otherwise, if $S'(z_r) = 0$ then repeated differentiation of (1) would show that S is identically zero. Szegő [?] showed that when the α_i's are real, as for the equation (2) we are considering, Stieltjes polynomials cannot have zeros at any of the α_i's. Thus, when the $\alpha_1 < \alpha_2 < \alpha_3$ are real, the zeros of Stieltjes polynomials are simple and real, and lie in $(\alpha_1, \alpha_3) \setminus \{\alpha_2\}$.

Secondly, we will use two results due to Shah [?]. For one, no Van Vleck zero is a zero of the corresponding Stieltjes polynomial. And, between any zero of a Stieltjes polynomial and a zero of a corresponding Van Vleck polynomial there is either a zero of the derivative of the Stieltjes polynomial or a singular point α_i.

Finally, we note a result originally proved by Van Vleck [?] in the real case, and extended to the complex case by Marden [?] that every zero of the Van Vleck polynomials $V(z)$ in (1) lies in the smallest convex polygon containing $\alpha_1, \ldots, \alpha_n$. For the equation under consideration here (2), every ν for which (2) admits a polynomials solution lies in (α_1, α_3).

We now proceed to the lemma, which is completely general for all $\rho_i > 0$ and $\alpha_1 < \alpha_2 < \alpha_3$. The equation (2) is invariant under affine transformations $x \mapsto ax + b$, so for convenience we assume that $\alpha_1 < 0 = \alpha_2 < \alpha_3$.

Lemma 2.1. *Let k be a positive integer. Then no Van Vleck zero of order k is a Van Vleck zero of order $k+1$.*

Proof. Suppose there is a ν such that S_k and S_{k+1} are Stieltjes polynomials of degree k and $k+1$, respectively, corresponding to Van Vleck polynomials with a zero at ν. Then $\nu \in (\alpha_1, \alpha_3)$, and since (2) is invariant under $x \mapsto -x$, we may assume that $\nu \in [0, \alpha_3)$. Employing the results noted above, the zeros of S_k and S_{k+1} are all simple and in $(\alpha_1, \alpha_3) \setminus \{0\}$. Since between any zero of S_i ($i \in \{k, k+1\}$) and ν there is either a zero of S_i' or 0, all of the zeros of S_k and S_{k+1} lie in the union $(\alpha_1, 0) \cup (\nu, \alpha_3)$. Moreover, if consecutive zeros of S_i bracket the interval $[0, \nu]$, then there is a zero of the derivative of S_i between ν and the larger of the two zeros.

If S is a Stieltjes polynomial of degree j, then substitution into (2) and identification of powers of x implies that

$$\mu = \mu_j = j\left(j - 1 + 2(\rho_1 + \rho_2 + \rho_3)\right). \tag{4}$$

Thus S_k and S_{k+1} satisfy

$$\begin{aligned} A(x)S_k'' + B(x)S_k' &= \mu_k(x-\nu)S_k && (5)\\ A(x)S_{k+1}'' + B(x)S_{k+1}' &= \mu_{k+1}(x-\nu)S_{k+1} && (6) \end{aligned}$$

Define the integrating factor

$$J(x) = \prod_{i=1}^{3} |x - \alpha_i|^{2\rho_i}.$$

Then $J' = J\frac{B}{A}$. We derive two expressions, the first of which is obtained by multiplying (5) by $\mu_{k+1}S_{k+1}$ and (6) by $\mu_k S_k$ and taking the difference. The second is obtained by dividing the equations (5 - 6) by A, multiplying (5) by S_{k+1}, (6) by S_k, and taking the difference of the result. The result is the following:

$$\begin{aligned} \frac{d}{dx}\left[J\left(\mu_{k+1}S_k'S_{k+1} - \mu_k S_k S_{k+1}'\right)\right] &= (\mu_{k+1} - \mu_k)\, J S_k' S_{k+1}', && (7)\\ \text{and} \qquad \frac{d}{dx}\left[J\left(S_{k+1}'S_k - S_{k+1}S_k'\right)\right] &= (\mu_{k+1} - \mu_k)\, Q S_k S_{k+1}, && (8) \end{aligned}$$

where $Q(x) = (x-\nu)J(x)/A(x)$. Moreover, at any of the singular points $\alpha_1, 0$, or α_3,

$$\mu_{k+1}S_k'S_{k+1} - \mu_k S_k S_{k+1}' = 0. \tag{9}$$

Notice that $Q(x) < 0$ for all $x \in (\alpha_1, 0) \cup (\nu, \alpha_3)$. Now, consider two consecutive zeros of S_k, $x_1 < x_2$ in either $(\alpha_1, 0)$ or (ν, α_3). Suppose that S_k and S_{k+1} are positive in (x_1, x_2). Then

$$J\left(S_{k+1}'S_k - S_{k+1}S_k'\right)\big|_{x=x_1} \le 0 \quad \text{and} \quad J\left(S_{k+1}'S_k - S_{k+1}S_k'\right)\big|_{x=x_2} \ge 0,$$

but, according to (8), $J\left(S_{k+1}'S_k - S_{k+1}S_k'\right)$ must be strictly decreasing on (x_1, x_2), which is a contradiction. Thus, S_{k+1} must change sign on (x_1, x_2).

Since $J(\alpha_i) = 0$, a similar argument shows that there is a zero of S_{k+1} (i) between any two zeros of S_k, (ii) between α_1 and the smallest zero of S_k in $(\alpha_1, 0)$, (iii) between the largest zero of S_k in $(\alpha_1, 0)$ and 0, and (iv) between the largest zero of S_k in $(0, \alpha_3)$ and α_3.

There are two cases to consider, depending on whether there is a zero of S_k less than zero.

Case 1: There is a zero of S_k in $(\alpha_1, 0)$. Firstly, if all the zeros of S_k (and hence all the zeros of S_{k+1}) were to lie in $(\alpha_1, 0)$, then $JS_k'S_{k+1}'$ would not change sign in $(0, \alpha_3)$, which would contradict (7), since $J(0) = J(\alpha_3) = 0$.

So, there must be at least one zero of S_k in (ν, α_3). A zero of S_k in $(\alpha_1, 0)$ implies that the zeros of S_k and S_{k+1} interlace, and hence that the zeros of the derivatives of S_k and S_{k+1} interlace [**?**]. It is possible for a Van Vleck zero to equal zero, but since $A(0) = 0$, this is possible if and only if the derivative of the corresponding Stieltjes polynomial has a zero at zero. But, since the zeros of S_k' and S_{k+1}' interlace in this case, it is impossible for $\nu = 0$ to be a Van Vleck zero corresponding to Stieltjes polynomials of successive orders.

Thus, it must be that $\nu > 0$ and $S_k'(\nu), S_{k+1}'(\nu) \neq 0$. And since the zeros of S_k' and S_{k+1}' interlace, the smallest zero of S_k' in (ν, α_3) is less than the smallest zero of S_{k+1}' in this interval. Let ξ be the smallest zero of S_k' in (ν, α_3). We may assume that $S_k, S_{k+1} > 0$ in $[0, \nu]$, so that $S_{k+1}' > 0$ in $[0, \xi]$, and hence

$$\left(\mu_{k+1}S_k'S_{k+1} - \mu_k S_k S_{k+1}'\right)\big|_{x=\xi} = -\mu_k S_k(\xi)S_{k+1}'(\xi) < 0. \tag{10}$$

But this, along with (9), contradicts (7) since $JS_k'S_{k+1}' > 0$ in $(0, \xi)$.

Case 2: All the zeros of S_k lie in (ν, α_3). In this case, there can be at most one zero of S_{k+1} in $(\alpha_1, 0)$. If there is one such zero of S_{k+1} in $(\alpha_1, 0)$, then there is a zero of S_{k+1}' between ν and the smallest zero of S_{k+1} in (ν, α_3). Thus, whether there is one or no zeros of S_{k+1} in $(\alpha_1, 0)$, $JS_k'S_{k+1}'$ does not change sign in $(\alpha_1, 0)$. But this contradicts (7) since $J(\alpha_1) = J(0) = 0$. ☐

Now we are in a position to prove our main result.

Theorem 2.1. *Let k be a positive integer. Write the Van Vleck zeros of order k in increasing order as $\nu_1^{(k)} < \nu_2^{(k)} < \cdots < \nu_{k+1}^{(k)}$. Then*

$$\alpha_1 < \nu_1^{(k+1)} < \nu_1^{(k)} < \nu_2^{(k+1)} < \nu_2^{(k)} < \cdots < \nu_{k+1}^{(k)} < \nu_{k+2}^{(k+1)} < \alpha_3. \tag{11}$$

In other words, the Van Vleck zeros of order k and $k+1$ interlace: between any two Van Vleck zeros of order k there is a Van Vleck zero of order $k+1$, and vice versa.

Proof. Under the transformation $x \mapsto (x-\alpha_2)/(\alpha_2-\alpha_1)$, $\nu \mapsto (\nu-\alpha_2)/(\alpha_2-\alpha_1)$ the polynomial coefficients $A(x)$ and $B(x)$ in the Lamé equation (2) take the form

$$\begin{aligned} A(x) &= x(x+1)(x-\alpha), \\ B(x) &= 2(\rho_1+\rho_2+\rho_3)x^2 + 2\left(\rho_2+\rho_3-\alpha(\rho_1+\rho_2)\right)x - 2\alpha\rho_2, \end{aligned}$$

where $\alpha = (\alpha_3 - \alpha_2)/(\alpha_2 - \alpha_1)$. Now, suppose $S_k(x) = \sum_{j=0}^{k} a_j x^j$ is a degree k polynomial solution of (2). Substitution into (2) and identification of the powers of x yields the following relation for $j = 0, \ldots, k$:

$$(\mu_{j-1} - \mu_k)a_{j-1} + j\left[(1-\alpha)(j-1) + 2\left(\rho_2 + \rho_3 - \alpha(\rho_1 + \rho_2)\right)\right]a_j \\ -(j+1)\alpha\left(j + 2\rho_2\right)a_{j+1} = -\mu_k \nu a_j, \qquad (12)$$

where μ_k is as in (4) and $a_{-1} = a_{k+1} = 0$. Therefore, the coefficients of S_k and the Van Vleck zeros of order k are the eigenvectors and eigenvalues, respectively, of a matrix $B^{(k)}$, whose coefficients are functions of ρ_1, ρ_2, ρ_3, and α.

Since the eigenvalues of a matrix are continuous functions of its entries [**?**], and in light of Lemma 2.1, it suffices to show that for some particular values of ρ_1, ρ_2, ρ_3, and $\alpha > 0$, the eigenvalues of $B^{(k)}$ and $B^{(k+1)}$ interlace. So let $\rho_1 = \rho_2 = \rho_3 = 1/2$. Then $B^{(k)}$ is tridiagonal, with nonzero entries given by

$$b_{j,j-1}^{(k)} = 1 - \frac{j(j-2)}{k(k+2)}, \quad b_{j,j}^{(k)} = (\alpha - 1)\frac{j(j-1)}{k(k+2)}, \quad b_{j,j+1}^{(k)} = \alpha\frac{j^2}{k(k+2)} \qquad (13)$$

$B^{(k)}$ is thus a function of α, which we write as

$$B^{(k)}(\alpha) = B^{(k)}(0) + \alpha A^{(k)}.$$

Note that $B^{(k)}(0)$ is bidiagonal, so its eigenvalues are given by the diagonal entries $-j(j-1)/k(k+2)$. Since

$$\frac{j(j-1)}{(k+1)(k+3)} < \frac{j(j-1)}{k(k+2)} < \frac{j(j+1)}{(k+1)(k+3)}, \quad j = 2, 3, \ldots, k+1,$$

the eigenvalues of $B^{(k)}(0)$ and $B^{(k+1)}(0)$ interlace except for a common eigenvalue at zero. Call those eigenvalues of $B^{(k)}$ and $B^{(k+1)}$ that are zero at $\alpha = 0$, $\lambda_0^{(k)}$ and $\lambda_0^{(k+1)}$, respectively. These are functions of α, and $\lambda_0^{(k)}(\alpha) = \nu_{k+1}^{(k)}$ when $\alpha > 0$. Therefore, for small $\alpha > 0$, the eigenvalues of $B^{(k)}$ and $B^{(k+1)}$ interlace as long as

$$\left.\frac{d}{d\alpha}\lambda_0^{(k)}\right|_{\alpha=0} < \left.\frac{d}{d\alpha}\lambda_0^{(k+1)}\right|_{\alpha=0}. \qquad (14)$$

The rate of change of the eigenvalues at $\alpha = 0$ are

$$\left.\frac{d}{d\alpha}\lambda_0^{(k)}\right|_{\alpha=0} = \frac{\mathbf{w}^T A^{(k)}\mathbf{v}}{\mathbf{w}^T\mathbf{v}},$$

where $\mathbf{w}^T$ and $\mathbf{v}$ are left and right eigenvectors of $B^{(k)}(0)$ associated with the zero eigenvalue, and similarly for $\lambda_0^{(k+1)}$. Since the entries of $B^{(k)}(0)$ in the first row are zero, the left eigenvector associated with the zero eigenvalue is $\mathbf{w} = (1, 0, \ldots, 0)^T$. A simple calculation shows that the ratio of the second and first entries of $\mathbf{v}$ satisfy $v_2/v_1 = -b_{21}^{(k)}(0)/b_{22}^{(k)}(0)$. The only nonzero entry of

$A^{(k)}$ in the first row is $a_{12}^{(k)}$, so

$$\begin{aligned}\left.\frac{d}{d\alpha}\lambda_0^{(k)}\right|_{\alpha=0} &= a_{12}^{(k)}\frac{v_2}{v_1} = -a_{12}^{(k)}\frac{b_{21}^{(k)}(0)}{b_{22}^{(k)}(0)} = \frac{k+2}{k+3} \\ &< \frac{k+3}{k+4} = \left.\frac{d}{d\alpha}\lambda_0^{(k+1)}\right|_{\alpha=0}\end{aligned} \tag{15}$$

Therefore, for small $\alpha > 0$ and $\rho_i = 1/2$, the eigenvalues of $B^{(k)}$ and $B^{(k+1)}$ interlace. Thus, the Van Vleck zeros of order k and $k+1$ must interlace for any fixed positive ρ_i's and any $\alpha_1 < \alpha_2 < \alpha_3$. For, if there were a set of ρ_i's and α_i's for which the interlacing (11) did not hold, the continuity of the eigenvalues of a matrix with respect to its entries and the intermediate value theorem would imply the existence of a set of ρ_i, α_i for which $B^{(k)}$ and $B^{(k+1)}$ had a common eigenvalue, contradicting Lemma 2.1. □

3 The non-orthogonality of the spectral polynomials

The construction above results in a six parameter family of matrices $\{B^{(k)}\}_{k=1}^{\infty}$ for which the eigenvalues for successive k's interlace and are in the interval (α_1, α_3). Note that $B^{(k)}$ is not a submatrix of $B^{(k+1)}$, so this result is not a simple consequence of the Cauchy interlacing theorem. Moreover, consider the spectral polynomials of $B^{(k)}$ formed from the Van Vleck zeros of order k:

$$p_{k+1}(x) = \prod_{i=1}^{k+1}\left(x - \nu_i^{(k)}\right). \tag{16}$$

Then p_k is a polynomial of degree k with simple zeros that interlace with the zeros of p_{k+1}. In this section we show that the family $\{p_k\}$ is not orthogonal with respect to any measure. The theorem depends on the following technical lemma.

Lemma 3.1. *For any distinct numbers $\alpha_1, \alpha_2, \alpha_3$ and any $\rho_1, \rho_2, \rho_3 > 0$,*

$$\sum_{i=1}^{k+1}\nu_i^{(k)} = \frac{\alpha_1+\alpha_2+\alpha_3}{3}k + C_1(\alpha_j,\rho_j) + \mathcal{O}\left(\frac{1}{k}\right), \quad \textit{and} \tag{17}$$

$$\begin{aligned}\sum_{i=1}^{k+1}\left(\nu_i^{(k)}\right)^2 &= \left[\frac{\alpha_1^2+\alpha_2^2+\alpha_3^2}{5} + \frac{2}{15}(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3)\right]k \\ &\quad + C_2(\alpha_j,\rho_j) + \mathcal{O}\left(\frac{1}{k}\right)\end{aligned} \tag{18}$$

where C_1 and C_2 depend only on the α_j's and ρ_j's.

Proof. We make the same transformation as in the proof of Theorem 11. From (12) we see that the transformed Van Vleck zeros are the eigenvalues of the tridiagonal matrix $B^{(k)}$ whose non-zero entries are

$$b^{(k)}_{j,j-1} = 1-\frac{\mu_{j-2}}{\mu_k}, \quad b^{(k)}_{j,j} = \frac{(j-1)\left((\alpha-1)(j-2)+g_1\right)}{\mu_k}, \quad b^{(k)}_{j,j+1} = \alpha\frac{j(j+2\rho_2)}{\mu_k},$$

where $g_1 = -2(\rho_2+\rho_3-\alpha(\rho_1+\rho_2))$. The first equality (17) follows from a calculation of the trace:

$$\begin{aligned}
\mathrm{tr}\left(B^{(k)}\right) &= \frac{(k+1)k(k-1)(\alpha-1)/3 + k(k+1)g_1/2}{k\left(k-1+2(\rho_1+\rho_2+\rho_3)\right)} \\
&= \frac{\alpha-1}{3}(k+1) + \frac{\alpha-1}{3}(1-2(\rho_1+\rho_2+\rho_3)) + \frac{g_1}{2} + \mathcal{O}\left(\frac{1}{k}\right)
\end{aligned}$$

Making the transformation back to the original variables, $\nu \mapsto \nu(\alpha_2-\alpha_2)+\alpha_2$,

$$\begin{aligned}
\sum_{i=1}^{k+1} \nu_i^{(k)} &= (\alpha_2-\alpha_1)\,\mathrm{tr}\left(B^{(k)}\right) + \alpha_2(k+1) \\
&= \left[(\alpha_2-\alpha_1)\frac{\alpha-1}{3}+\alpha_2\right](k+1) \\
&\quad +(\alpha_2-\alpha_1)\left[\frac{(\alpha-1)(1-2(\rho_1+\rho_2+\rho_3))}{3}+\frac{g_1}{2}\right] + \mathcal{O}\left(\frac{1}{k}\right),
\end{aligned}$$

and since $(\alpha_2-\alpha_1)(\alpha-1)/3+\alpha_2 = (\alpha_1+\alpha_2+\alpha_3)/3$, this establishes (17).

To prove (18) we must compute the trace of $\left(B^{(k)}\right)^2$. The diagonal terms of this matrix are given by

$$\begin{aligned}
\left(B^{(k)2}\right)_{jj} &= \{(\mu_k-\mu_{j-2})\,\alpha(j-1)(j-1+2\rho_2) \\
&\quad + (j-1)^2\left((\alpha-1)(j-2)+g_1\right)^2 + (\mu_k-\mu_{j-1})\,\alpha j(j+2\rho_2)\Big\}/\mu_k^2 \\
&= \frac{\left[(\alpha-1)^2-2\alpha\right]j^4+\mathcal{O}(j^3)}{k^4+\mathcal{O}(k^3)} + \frac{2\alpha j^2+\mathcal{O}(j)}{k^2+\mathcal{O}(k)}
\end{aligned}$$

Since the only terms that are not constant (w.r.t. k) or $\mathcal{O}(1/k)$ come from the j^4/k^4 and j^2/k^2 terms,

$$\mathrm{tr}\left(B^{(k)2}\right) = \left[\frac{(\alpha-1)^2}{5}+\frac{4\alpha}{15}\right]k + \text{const.} + \mathcal{O}\left(\frac{1}{k}\right).$$

As before, we transform back to the original variables,

$$\begin{aligned}
\sum_{i=1}^{k+1}\left(\nu_i^{(k)}\right)^2 &= (\alpha_2-\alpha_1)^2\,\mathrm{tr}\left(B^{(k)2}\right) + 2\alpha_2(\alpha_2-\alpha_1)\,\mathrm{tr}\left(B^{(k)}\right) + \alpha_2^2(k+1) \\
&= \left\{(\alpha_2-\alpha_1)^2\left[\frac{(\alpha-1)^2}{5}+\frac{4\alpha}{15}\right] + 2\alpha_2(\alpha_2-\alpha_1)\frac{\alpha-1}{3}+\alpha_2^2\right\}k \\
&\quad + C_2(\alpha_j,\rho_j) + \mathcal{O}\left(\frac{1}{k}\right).
\end{aligned}$$

A simplification of the quantity in braces above establishes (18) $\square$

Theorem 3.1. *The spectral polynomials* (16) *are not orthogonal with respect to any measure.*

Proof. Suppose that $\{p_k\}$ is orthogonal with respect to some measure. Then the polynomials must satisfy a three-term recurrence relation of the following form [**?**]. There exists sequences $\{a_n\}$ and $\{b_n\}$, with $a_n \in \mathbb{R}$ and $b_n > 0$ such that

$$p_n(x) = (x - a_n)\, p_{n-1}(x) - b_n\, p_{n-2}(x). \tag{19}$$

We will show that if the polynomials defined by (16) satisfy the relation (19), then a_n converges to a real number and b_n converges to a positive number. This implies, by a well known theorem, that the density of zeros of p_n in the limit as $n \to \infty$ is described by a measure which is different from the asymptotic density of zeros of spectral polynomials calculated by Borcea and Shapiro [**?**]. This contradiction will imply the truth of the theorem.

First, identifying the coefficients of x^n in both sides of (19), yields the equation

$$a_n = \sum_i \nu_i^{(n-1)} - \sum_i \nu_i^{(n-2)}.$$

Therefore, Lemma 3.1 implies

$$\lim_{n\to\infty} a_n = \frac{\alpha_1 + \alpha_2 + \alpha_3}{3}. \tag{20}$$

Next, we consider the coefficients of x^{n-1}. We get

$$\begin{aligned} b_n &= \left(\sum_i \nu_i^{(n-1)} - \sum_i \nu_i^{(n-2)}\right) \sum_i \nu_i^{(n-2)} + \sum_{i<j} \nu_i^{(n-2)} \nu_j^{(n-2)} - \sum_{i<j} \nu_i^{(n-1)} \nu_j^{(n-1)} \\ &= -\frac{1}{2}\left(\sum_i \nu_i^{(n-1)} - \sum_i \nu_i^{(n-2)}\right)^2 + \frac{1}{2}\left[\sum_i \left(\nu_i^{(n-1)}\right)^2 - \sum_i \left(\nu_i^{(n-2)}\right)^2\right] \\ &= -\frac{1}{2} a_n^2 + \frac{1}{2}\left[\sum_i \left(\nu_i^{(n-1)}\right)^2 - \sum_i \left(\nu_i^{(n-2)}\right)^2\right] \end{aligned} \tag{21}$$

Utilizing the second equality in Lemma 3.1 and combining the result with (20) and (21), we find

$$\lim_{n\to\infty} b_n = \frac{2}{45}\left(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \alpha_1\alpha_2 - \alpha_2\alpha_3 - \alpha_1\alpha_3\right),$$

which is positive for all real $\alpha_1 < \alpha_2 < \alpha_3$.

Now, since $a_n \to a \in \mathbb{R}$ and $b_n \to b \in (0, \infty)$, according to Theorem 5.3 of [**?**], the polynomials p_n have the asymptotic zero distribution $\omega_{[\alpha,\beta]}$ with density

$$\frac{d\omega_{[\alpha,\beta]}(x)}{dx} = \begin{cases} \frac{1}{\pi\sqrt{(\beta - x)(x - \alpha)}}, & \text{if } \quad x \in (\alpha, \beta) \\ 0 & \text{elsewhere,} \end{cases} \tag{22}$$

where $\alpha = a - 2/b$ and $\beta = a + 2/b$. However, in [?] it is shown that the spectral polynomials p_n defined by (16) have the asymptotic zero distribution given by a probability measure supported on (α_1, α_3), with density $\rho_A(x)$ given by

$$\rho_A(x) = \begin{cases} \frac{1}{2\pi}\int_{\alpha_2}^{\alpha_3} \frac{ds}{\sqrt{(\alpha_3-s)(s-\alpha_2)(s-\alpha_1)(s-x)}} & \text{if} \quad \alpha_1 < x < \alpha_2, \\ \frac{1}{2\pi}\int_{\alpha_1}^{\alpha_2} \frac{ds}{\sqrt{(\alpha_3-s)(\alpha_2-s)(s-\alpha_1)(x-s)}} & \text{if} \quad \alpha_2 < x < \alpha_3. \end{cases} \tag{23}$$

Since the limiting distributions in (22) and (23) are unequal, it cannot be that the spectral polynomials obey the recurrence relation (19), and hence they are not orthogonal with respect to any measure. $\square$

4 The Lamé equation

The most common form of the Lamé equation in the literature is

$$\frac{d^2\phi}{dx^2} + (n(n+1)k^2\mathrm{sn}^2\,(x,k) - h)\phi = 0 \tag{24}$$

where $\mathrm{sn}\,(x,k)$ is the Jacobi elliptic function with modulus k, $0 < k < 1$. For fixed k and n, we say that h is an eigenvalue if (24) admits a nontrivial solution. It is well known that if we assume n to be a positive integer (as we do from now on), then (24) has exactly $2n+1$ distinct eigenvalues. Furthermore, the corresponding eigenfunctions are the Lamé functions of the first kind:

$$\phi^{\gamma_1,\gamma_2,\gamma_3}(x) = \mathrm{sn}^{\gamma_1}\,(x,k)\ \mathrm{cn}^{\gamma_2}\,(x,k)\ \mathrm{dn}^{\gamma_3}\,(x,k)\ P_m(\mathrm{sn}^2(x,k)) \tag{25}$$

where $\gamma_i \in \{0,1\}$, and P_m is a polynomial of degree m with $n = 2m + |\gamma|$.

The set Λ_n of eigenvalues can be divided into eight disjoint subsets according to the different values of γ_i, namely

$$\Lambda_n = \begin{cases} \Lambda_n^{0,0,0} \cup \Lambda_n^{1,1,0} \cup \Lambda_n^{1,0,1} \cup \Lambda_n^{0,1,1} & \text{if } n \text{ is even} \\ \Lambda_n^{1,0,0} \cup \Lambda_n^{1,0,0} \cup \Lambda_n^{0,0,1} \cup \Lambda_n^{1,1,1} & \text{if } n \text{ is odd} \end{cases}$$

where $\Lambda_n^{\gamma_1,\gamma_2,\gamma_3}$ is the set of all eigenvalues h having an eigenfunction of the form $\phi^{\gamma_1,\gamma_2,\gamma_3}$. The cardinality of each subset is

$$|\Lambda_n^{0,0,0}| = n/2 + 1,\ |\Lambda_n^{1,0,0}| = |\Lambda_n^{0,1,0}| = |\Lambda_n^{0,0,1}| = (n+1)/2,$$
$$|\Lambda_n^{1,1,0}| = |\Lambda_n^{1,0,1}| = |\Lambda_n^{0,1,1}| = n/2,\ |\Lambda_n^{1,1,1}| = (n-1)/2.$$

Note that for n even

$$|\Lambda_n| = |\Lambda_n^{0,0,0}| + |\Lambda_n^{1,1,0}| + |\Lambda_n^{1,0,1}| + |\Lambda_n^{0,1,1}| = 2n + 1$$

and similarly for n odd.

Using the theory of Hill's equation, Volkmer [?] recently obtained various interlacing properties satisfied by the eigenvalues for a fixed value of the parameter n. We will use Theorem 2.1 to give new interlacing results when n takes consecutive integer values.

First, we need to rewrite Lamé equation into its algebraic form. Making the substitution $x \mapsto \mathrm{sn}^2(x, k)$, we get

$$\frac{d^2\phi}{dx^2} + \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-k^{-2}}\right)\frac{d\phi}{dx} = \frac{n(n+1)x - k^{-2}h}{4x(x-1)(x-k^{-2})}\phi. \tag{26}$$

Now, if we substitute the corresponding Lamé function

$$\phi^{\gamma_1,\gamma_2,\gamma_3}(x) = |x|^{\gamma_1/2}|x-1|^{\gamma_2/2}|x-k^{-2}|^{\gamma_3/2}P_m(x)$$

into (26), then one can easily verify that the polynomial P_m satisfies the Heine-Stieltjes equation

$$\frac{d^2P_m}{dx^2} + \left(\frac{\gamma_1+1/2}{x} + \frac{\gamma_2+1/2}{x-1} + \frac{\gamma_3+1/2}{x-k^{-2}}\right)\frac{dP_m}{dx} = \frac{m(m+|\gamma|+1/2)(x-\lambda)}{x(x-1)(x-k^{-2})}P_m,$$

where $\lambda(\gamma, k)$ is related to the eigenvalue h through the relation

$$\lambda = \frac{k^{-2}h - (1+k^2)\gamma_1 - \gamma_2 - k^2\gamma_3 - 2k^2\gamma_1\gamma_3 - 2\gamma_1\gamma_2}{m(m+|\gamma|+1/2)}. \tag{27}$$

The next result is then an immediate consequence of Theorem 2.1.

Proposition 4.1. *For each $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \{0,1\}^3$, let $h^{\gamma}_{j,n}$ denote the ordered eigenvalues of (24) in Λ^{γ}_n, and let $\lambda^{\gamma}_{j,n}$ be defined by (27) with $h = h^{\gamma}_{j,n}$. We have*

$$0 < \lambda^{\gamma}_{j,n+2} < \lambda^{\gamma}_{j,n} < \lambda^{\gamma}_{j+1,n+2} < k^{-2}$$

for all $j = 1, ..., |\Lambda^{\gamma}_n|$.

5 Remarks and a conjecture

It is a well-known fact that the zeros of orthogonal polynomials interlace [?]. The family $\{p_k\}$ of spectral polynomials (16) is an example of a family of polynomials with interlacing zeros, but is not orthogonal, and is thus a counterexample to the converse of the statement that the zeros of orthogonal polynomials interlace. We are unaware of any other example of such a family arising from a second order differential equation. This is all the more striking when we consider that the density function ρ_A defined in (23) satisfies, by a theorem in [?], the following Heun differential equation:

$$8A(x)\rho_A''(x) + 8A'(x)\rho_A'(x) + A''(x)\rho_A(x) = 0,$$

where $A(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$ is the function used to define the Lamé equation (2). It would be interesting to determine if there are any other spectral polynomials of second order differential operators of this kind, i.e. with interlacing zeros, non-orthogonality and with asymptotic density satisfying a Heun equation.

In [?] we considered the Lamé equation in the case when $\alpha_1, \alpha_2, \alpha_3$ are the vertices of an equilateral triangle in the complex plane. Since the Lamé equation is invariant under complex affine transformations, we may assume in this case that the α_i's are the third roots of unity, $\alpha_j = \exp(i(j-1)2\pi/3)$, $j = 1, 2, 3$. In the special case when $\rho_1 = \rho_2 = \rho_3$, we found that the Van Vleck zeros of order $3k - 1$ are of the form

$$\lambda_n \, e^{\frac{2\pi i}{3} j}, \quad n = 1, \ldots, k, \quad j = 0, 1, 2,$$

where $\lambda_n \in (0, 1)$ is real. In other words, the Van Vleck zeros lie on the lines connecting the triangle incenter to its vertices. When the order is $3k$ or $3k+1$, there is an additional Van Vleck zero at the center of the triangle. Numerical evidence suggests an analog to Theorem 2.1 in the complex case. Let $\lambda_n^{(3k-1)}$, $n = 1, \ldots, k$ be the distance of the Van Vleck zeros of order $3k - 1$ from the triangle incenter. These are distinct. If we label them in increasing order as $\lambda_1^{(3k-1)} < \lambda_2^{(3k-1)} < \cdots < \lambda_k^{(3k-1)}$, we conjecture that these distances interlace with those of order $3k + 2$:

$$0 < \lambda_1^{(3k+2)} < \lambda_1^{(3k-1)} < \lambda_2^{(3k+2)} < \lambda_2^{(3k-1)} < \cdots < \lambda_k^{(3k+2)} < \lambda_k^{(3k-1)} < \lambda_{k+1}^{(3k+2)} < 1.$$

We note that, unlike in the real case this property only holds in the complex case when the ρ_i's are all equal, since when the ρ_i's are not all equal, the symmetry breaks and the Van Vleck zeros do not lie on the lines connecting the incenter with the vertices.